$\overline{\eta}_{ot}$ = instantaneous overall effectiveness factor given by Eq. 12
θ = dimensionless time = $(k_d \cdot t)$
ξ = dimensionless radial position = r/R
ρ_p = density of catalyst particle, g/cc
τ = space time, (s)(g cat.)/(cc)
ϕ = dimensionless group = $(3h)$
ϕ' = dimensionless group = $(3hA)$ = $(\phi \cdot A)$
ψ = dimensionless concentration = (C_r/C_R)

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Manuscript received January 13, 1978; revision received July 14, and accepted July 16, 1980.

Design and Experimental Evaluation of Controllers for Process Undisturbability

Selected state and/or output variables can be made undisturbable, i.e., invariant, to arbitrary, unmeasured changes in specific input variables by properly designed feedback and feedforward controllers. Simulation and experimental applications to a computer-controlled, pilot plant evaporator gave results superior to conventional controllers.

Necessary and sufficient conditions for undisturbability are expressed in terms of the structure of the coefficient matrices of the state space model and equivalently of the corresponding eigenvector matrix. The design procedure normally includes arbitrary specification of all closed-loop eigenvalues and up to r elements of each eigenvector, where r is the number of control (manipulated) variables.

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SCOPE

This paper describes a design procedure for process controllers that makes selected state and/or output variables *undisturbable*, i.e., invariant, to arbitrary changes in specified input variables. Note that if all process outputs are made undisturbable, the process can be said to exhibit "perfect" regulatory control. Thus, this approach should be of interest in a wide variety of process applications.

The first step was to derive the necessary and sufficient conditions to make selected state or output variables undisturbable with respect to specified input variable(s). The next step was to determine what systems can be made undisturbable by the use of *constant* feedforward and/or feedback controllers. The final step was to develop a practical design procedure and to evaluate the resulting controllers experimentally.

Other authors have dealt with *rejection* of disturbances before they reach the system outputs; *invariance* to external disturbances; and *localization* of the disturbances so they do not affect the system outputs. These concepts are similar to *undisturbability* as discussed in this paper, but are less rigorously defined and include significant differences. For example, disturbance localization is used in the context of a geometric interpretation. Therefore, this work includes a precise definition of *undisturbability* and points out some of the parallels and differences with familiar concepts such as *uncontrollability* and *unobservability*.

Necessary and sufficient conditions, in geometric terms, for the existence of state feedback controllers that localize disturbances from the system outputs have been presented by Wonham and Morse (1970), Bhattacharyya (1974), and the authors (Shah et al. 1974, 1977). Necessary and sufficient conditions for simultaneous disturbance localization plus setpoint/output decoupling have been reported by Fabian and Wonham (1975) and Chang and Rhodes (1975). Since the necessary and sufficient conditions were already known, this work focused on developing equivalent conditions in state-space terms that could be more easily applied, and which would lead to a practical, easily implementable design procedure. In contrast to the synthesis results of Wonham and Morse (1970), the emphasis here is on the use and application of structural results of Shah et al. (1977) to design controllers for undisturbability. New theoretical results on the design for undisturbability using proportional plus integral feedback are also included.

Also since to the best of the author's knowledge this design approach has never been evaluated *experimentally*, one of the prime objectives was to apply it to the computer-controlled, pilot-plant evaporator at the University of Alberta.

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CONCLUSIONS AND SIGNIFICANCE

Necessary and sufficient conditions under which specific state and/or output variables are undisturbable with respect to specified input, variables are developed in terms of the structure of the coefficient matrices of the state space model and in terms of the structure of the system eigenvector matrix. Thus, testing for the undisturbability of specific input-output pairs can be done by *inspection* of the appropriate matrices, and design accomplished by specifying controllers that produce the required structure in the closed-loop model.

These results led to a constructive design procedure for the required controllers. The design procedure normally permits the specification of arbitrary values for all system eigenvalues plus up to r elements of each eigenvector. Thus, under the conditions specified, system stability can be guaranteed and the general response characteristics influenced by the values specified for the eigenvalues and eigenvectors. An important characteristic of this design procedure is that it makes use of the information contained in the model about the effect of disturbances, whereas most other formal design schemes for feedback controllers do not. The design method also provides considerable insight into the system performance.

Undisturbability is closely related to the system property of uncontrollability, but is *not* identical. For example, a particular output is not *undisturbable*, if there is a signal path connected to the input variable of interest. However, the output may be uncontrollable by the same input. Undisturbability can also be explained in terms of signal flow graphs, frequency domain concepts and related to the concept of structural controllability (Lin, 1974). The controllers designed to produce output undisturbability can be interpreted as modifying the closed-loop system so that disturbance(s) affect only the unobservable modes, i.e., those that do not contribute to the output variables of interest.

To compensate for the effect of model inaccuracies or sustained unknown disturbances, the introduction of integral state feedback of the undisturbable state variables has been proposed. Such an approach is shown to be false by new results which identify particular rank deficiencies that arise when any eigenvalue/eigenvector assignment algorithm is used to design for integral state feedback of the undisturbable state variables.

Experimental results from the computer-controlled pilot-plant evaporator at the University of Alberta show that the controllers: produced the expected "undisturbability;" provided better control than conventional single variable controllers; and were comparable to controllers designed using other modern multivariable control techniques such as "optimal-quadratic control." They are particularly appropriate where the main objective is to eliminate the effect of a specific input on a particular state or output variable.

DISCUSSION

Before discussing the design method and applications, it is desirable to define *undisturbability* more precisely and to summarize the relevant theory. This will be done as concisely as possible, in a series of subsections, to provide the reader with an overview of the basic concepts.

Open-Loop Model. Like most modern control techniques, undisturbability is based on a mathematical model of the system. It is assumed that the system to be controlled can be adequately represented over the region of interest, by a linear, time-invariant, completely controllable, state-space model of the form:

$$\dot{x} = Ax + [B \;\; D]\left[\frac{u}{\xi}\right] \qquad y = Cx \tag{1}$$

The input vector in Eq. 1 has been partitioned into two parts: first the manipulated variables u; and secondly, the disturbance variables, ξ. In many applications this division is obvious from physical considerations, while in others it is one of the decisions to be made by the design engineer.

Control Law. It is assumed here that the control action can be represented by the following feedforward-feedback relationship:

$$u(t) = Kx(t) + K^{FF}\,\xi(t) \tag{2}$$

Note that when $K^{FF} = 0$, $C = I$, and K is diagonal, Eq. 2 represents conventional, multiloop, proportional, feedback control.

Closed-Loop Model. Combining Eqs. 1 and 2 leads directly to the closed-loop system:

$$\dot{x}(t) = Hx(t) + L\,\xi(t) \tag{3}$$
$$y(t) = Cx(t)$$

where $\quad H = A + BK$ and $L = D + BK^{FF}$

Eq. 1 can be regarded as a special case of Eq. 3 in which $K = 0$ and $K^{FF} = 0$ so that derivations based on Eq. 3 apply to both closed-loop and open-loop systems. Similarly, the design procedure can be easily extended to apply to discrete systems, as illustrated in the later applications.

Closed-Loop System Response. For a system described by Eq. 3, the effect of the j^{th} disturbance, $\xi_j(t)$, on the time domain response of the closed-loop system is given by:

$$y^{\xi}(t) = Cx^{\xi}(t) = C\int_0^t \exp\{H(t-\tau)\}l_j\xi_j(\tau)\,d\tau \tag{4}$$

where l_j is the j^{th} column of L and $j \in (1 \ldots q)$.

The discussion of undisturbability in this paper is based on Eq. 4 which is written in terms of a single disturbance. However, there is no loss of generality, because the effect of all disturbances $\{\xi_j(t),\ j = 1, \ldots q\}$, the effect of initial conditions, $x(0)$, and/or of external inputs can all be calculated independently and summed to give the total time domain response, $y(t)$.

Definition 1. State undisturbability of a system that can be represented by Eq. 3 is defined by Shah et al. (1977) as: the i^{th} state variable, x_i, of a system characterized by the pair (H, l_j) is undisturbable with respect to disturbance ξ_j, if for arbitrary $\xi_j(t)$, and for all $t>0$, x_i^{ξ}, the i^{th} element of x^{ξ}, satisfies $x_i^{\xi}(t) = 0$.

Although there is a close relationship, undisturbability is *not* identical with uncontrollability. In general, if x_i is undisturbable by ξ_j, it is also uncontrollable by ξ_j. However, disturbability does *not* always imply controllability, since a system may be disturbable by ξ_j but may not be controllable with respect to ξ_j as input. The relationship between undisturbability and uncontrollability is illustrated in the following applications and is discussed further by Shah et al. (1977).

Conditions for Undisturbability. Necessary and sufficient conditions for the existence of state feedback controllers that will decouple the effect of disturbances, $\xi(t)$, from outputs, $y(t)$, have been defined in *geometric* terms as follows:

1) the range space of D must lie within the maximal (A, B)—invariant subspace contained in Kernel C (Wonham and Morse, 1970; Bhattacharyya, 1974), or
2) the range space of D must lie within the space spanned by at least q of the closed-loop system eigenvectors (where q is the dimension of the disturbance vector, ξ), which in turn must be contained in Kernel C (Shah et al., 1974).

However, in this design approach, these conditions are not tested directly. Instead, it is shown that the necessary and sufficient conditions for state undisturbability can be expressed in terms of the structure of the system matrix, H, or equivalently, the structure of the closed-loop eigenvector matrix, W. Then, a design procedure is presented that generates the required controller matrices based on eigenvalues and elements of the closed-loop eigenvectors chosen by the designer.

BASIS FOR DESIGN

Inspection of Eqs. 3 and 4 shows that the response of the state variables of a system to a change in the j^{th} disturbance, is "characterized" by the pair H, l_j. Obviously, if $l_j = 0$, the system is undisturbable. However, for the more general case, where l_j is nonzero the conditions governing undisturbability can be summarized as follows (Shah et al. 1976).

A system characterized by the pair $\{H, l_j\}$ has $k<r$ undisturbable state variables $\{x_i, i=1,\ldots,k\}$ with respect to disturbance ξ_j, if and only if:

1) The matrix H, or equivalently the closed-loop eigenvector matrix W, is in the following form or can be brought into this form by a suitable reordering of the state variables:

$$H = \left[\begin{array}{c|c} H_1 & 0 \\ \hline H_3 & H_4 \end{array}\right] \qquad (5)$$

$$W = \left[\begin{array}{c|c} W_1 & 0 \\ \hline W_3 & W_4 \end{array}\right] \qquad (6)$$

and

2) The j^{th} column of L is of the form:

$$l_j = \left[\begin{array}{c} 0 \\ \hline l_{2j} \end{array}\right] \qquad (7)$$

where H and W are $n\times n$; H_1 and W_1 are $k\times k$; H_4 and W_4 are $(n-k)\times(n-k)$; l_j is $n\times 1$ and l_{2j} is $(n-k)\times 1$

The condition expressed in terms of the eigenvector matrix W assumes that the system has distinct eigenvalues. In closed-loop systems, this is not restrictive because in practical problems almost any gain matrix K will produce distinct eigenvalues and the design procedure described in the next section usually permits arbitrary specifications of all the eigenvalues. Proofs of the necessity and sufficiency of the conditions summarized in Eqs. 5 and 7 are presented elsewhere (Shah et al., 1977), and it is not difficult to show that Eq. 5 implies Eq. 6 and vice versa.

OUTPUT UNDISTURBABILITY

The undisturbability of the system output, y_i to disturbance, ξ_j, depends on c_i^T as well as on the pair $\{H, l_j\}$. This leads to the definition for output undisturbability by Shah et al. (1977): the i^{th} output, y_i, of a system characterized by $\{H, l_j, c_i^T\}$ is undisturbable with respect to disturbance, ξ_j, if for arbitrary $\xi_j(t)$, and for all $t > 0$, y_i^{ξ}, the i^{th} element of y^{ξ}, satisfies $y_i^{\xi}(t) = 0$.

Note that when system outputs are a subset of the state variables, conditions for state undisturbability can be directly applied to determine output undisturbability. However, when outputs are a linear combination of the state variables, the conditions for state undisturbability cannot be applied directly. However, it is still possible to design for output undisturbability by transforming the system before applying the conditions in Eqs. 5 to 7. The first step is to define a new vector, z, of dimension n, such that:

$$z = \left[\begin{array}{c} C \\ \hline E \end{array}\right] x \equiv Q^{-1} x \qquad (8)$$

where C is the $m\times n$ output matrix of the original system (assumed to be of full rank) and E is an $(n-m) \times (n-m)$ matrix chosen arbitrarily such that its $(n-m)$ rows are linearly independent of the rows of C. Under these conditions, Q exists; the original system defined by Eq. 1 becomes:

$$\dot{z} = Q^{-1} A\, Q\, z + Q^{-1} B\, u + Q^{-1} D\, \xi, \quad y = C\, Q\, z \qquad (9)$$

which can be rewritten as:

$$\dot{z} = \tilde{A}\, z + \tilde{B}\, u + \tilde{D}\, \xi, \quad y = \tilde{C}\, z \qquad (10)$$

The closed-loop system and feedforward matrices are appropriately represented by $\tilde{H}$ and $\tilde{L}$.

Since in the new canonical form $\{z_i = y_i,\ i \in 1 \ldots m\}$ the conditions for state undisturbability can be applied directly to the transformed system. Therefore, the same conditions provide a basis for the design of systems with undisturbable outputs. This is illustrated in one of the following examples.

Output undisturbability can also be easily expressed in terms of frequency domain concepts. The basic requirement is that the closed-loop transfer function $q_{ij}(s)$ between disturbance ξ_j and output y_i be identically zero. The frequency domain characterization of the disturbance rejection/minimization problem has been considered in detail elsewhere by Shah and Fisher (1978).

DESIGN PROCEDURE

The approach used here to design controllers that produce closed-loop systems with undisturbable state variables is as follows.

1) Use an eigenvalue/eigenvector assignment algorithm to generate a closed-loop eigenvector matrix, W, with the structure defined by Eq. 6.
2) Use feedforward control, if necessary, so that $l_j = Bk_j^{FF} + d_j$ has the structure defined by Eq. 7.
3) Calculate the feedback controller matrix K, that satisfies the relationship $H = A + BK = W \Lambda V$.

Eigenvalue/Eigenvector Assignment. The eigenvector assignment algorithm used in this work was the one developed by Srinathkumar and Rhoten (1975) which for most practical systems permits the arbitrary specification of all n eigenvalues plus up to r elements of each eigenvector. The algorithm is based on the following expansion of H:

$$H = A + BK = W\Lambda V = \left[\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right] \left[\begin{array}{c|c} \Lambda_1 & 0 \\ \hline 0 & \Lambda_2 \end{array}\right] \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{21} & V_{22} \end{array}\right] \qquad (11)$$

where $V \equiv W^{-1}$; W_{11}, Λ_1, and V_{11} are $r\times r$ and hence are not necessarily the same dimension as W_1 of Eq. 6. The expansion assumes that H has distinct eigenvalues which is the case in most practical problems.

If the first r rows of B form a nonsingular matrix, B_1, (which might require reordering of the state variables), a $r\times n$ feedback matrix, K, exists such that $r\times n$ elements of W_{11} and W_{12} can be chosen arbitrarily subject only to the requirement that $V = W^{-1}$ exists. Note that this arbitrary specification of the first r elements of each of the n eigenvectors requires only $(r\times n)-n$ degrees of freedom providing that at least one of r specified elements in each eigenvector is nonzero (Shah et al., 1975).

The remaining n degrees of freedom can be used to specify the closed-loop eigenvalues. Once W_{11}, W_{12}, Λ_1 and Λ_2 have been specified, W_{21} and W_{22} can be calculated from the following relationships which follow from Eq. 12:

$$W_{21}\, \Lambda_1 - PW_{21} = RW_{11} + SW_{11}\, \Lambda_1 \qquad (12)$$

$$W_{22}\, \Lambda_2 - PW_{22} = RW_{12} + SW_{12}\, \Lambda_2 \qquad (13)$$

where

$$\left.\begin{array}{l} S = B_2 B_1^{-1} \\ R = A_{21} - SA_{11} \\ P = A_{22} - SA_{12} \end{array}\right\} \qquad (14)$$

Eigenvalue-Eigenvector Assignment for State Undisturbability. If the number of desired *undisturbable* state variables, k, is less than the number of inputs, r, there are sufficient degrees of freedom, using the above procedure, to assign k elements of each eigenvector so that:

1) W has the structure defined by Eq. 6, and
2) all n closed-loop eigenvalues can be assigned which means that stability can be guaranteed.

If the number of undisturbable state variables, k, equals the number of inputs, r, the $r\times r$ elements of W_{12} in Eq. 11 can all be set to zero to meet the conditions for undisturbability required by Eq. 6. However, when $W_{12} = 0$, it follows from Eq. 13 that the $(n-r)$ eigenvalues of Λ_2 are the eigenvalues of P and thus cannot be assigned arbitrarily. The eigenvalues of P can be checked directly to see if they contribute to an unstable closed-loop system response. Necessary and sufficient conditions to produce $k = r$ undisturbable states and also guarantee closed-loop stability have been reported by Fabian and Wonham (1975) and Chang and Rhodes (1975). Once Λ, W_1 and $V = W^{-1}$ are known, the feedback matrix K can be calculated from:

$$K = B^* \, (W\Lambda V - A) \tag{15}$$

where $B^* = (B^TB)^{-1}B^T$, i.e., the pseudo-inverse of B. When W is obtained as outlined above using Eqs. 12 and 13, Eq. 15 yields an exact solution for K (Shah et al., 1975). In other cases, Eq. 15 gives a "least squares" approximate solution for K.

Note that in the eigenvalue/eigenvector assignment algorithm developed by Srinathkumar and Rhoten (1975), not all of the control objectives can be met if B_1 is singular. In such a case, the control objectives have to be modified such that the final B_1 obtained after re-ordering the state variables in nonsingular. However, Moore (1976) has proposed a method for eigenvalue/eigenvector assignment which does not require B_1 to be nonsingular and thus provides an important alternative to the approach used in this paper.

DESIGN FOR UNDISTURBABILITY USING PROPORTIONAL PLUS INTEGRAL FEEDBACK

Model inaccuracies and parameter variations occur in most practical applications and thus the original controller design objectives of undisturbability may not be realized. In such situations, the possibility exists that the introduction of integral feedback of the appropriate state variables would be desirable. The following analysis shows that independent integral state feedback of the undisturbable state variables is not possible.

The addition of integral feedback compensation will be considered by augmenting the original state vector $x(t)$ with a $p\times 1$ $(p \le k \le r)$ vector $x_I(t)$ where $x_I(t)$ is defined by:

$$\dot{x}_I(t) = T\, x(t) \tag{16}$$

In Eq. 16, T is a $p\times n$ matrix and it consists of p appropriate rows to include a subset of state variables or a linear combination of state variables requiring integral feedback. Note that because of controllability conditions on the augmented system, $p\le r$. Let $\hat{A}$ and $\hat{B}$ denote the augmented system state matrix and input matrix, respectively. Let the pair $(\hat{A},\hat{B})$ be first partitioned in the form:

$$\left[\begin{array}{c|cc} A_{11} & A_{12} & 0 \\ \hline A_{21} & A_{22} & 0 \\ T_1 & T_2 & 0 \end{array}\right] \quad \left[\begin{array}{c} B_1 \\ \hline B_2 \\ 0 \end{array}\right] \tag{17}$$

where A_{11} is a $k \times k$ matrix, A_{22} is a $(n-k)\times(n-k)$ matrix, T_1 is a $p\times k$ matrix, B_1 is a $k\times r$ matrix. If the state variables $x_1, x_2, \ldots x_k$ are to be made undisturbable, the structure of the closed-loop system matrix H and the corresponding eigenvector matrix W must be of the form (cf. Eqs. 5 and 6):

$$H = \left[\begin{array}{c|cc} H_{11} & 0 & 0 \\ \hline H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{array}\right] \leftrightarrow W = \left[\begin{array}{c|cc} W_{11} & 0 & 0 \\ \hline W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{array}\right] \tag{18}$$

If $K = [K_1\ K_2\ K_3]$ denotes the appropriately partitioned proportional plus integral state feedback matrix, it follows that for the case when $p=r$, no integral feedback can exist since Eq. 18 implies that $K_3=0$ (B_1 is assumed to be of full rank r). For $p<r$, the existence of integral feedback is not clearly obvious and this is considered next. It follows from Eqs. 17 and 18 that:

$$\begin{bmatrix} A_{22} + B_2K_2 & B_2\,K_3 \\ T_2 & 0 \end{bmatrix} = \begin{bmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{bmatrix} J_2 \begin{bmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{bmatrix}^{-1} \triangleq W_4 J_2 W_4^{-1} \tag{19}$$

This equation is a general one where the $(n+p-k)\times(n+p-k)$ Jordan block, J_2, may be specified (Klein and Moore, 1977), with the corresponding generalized eigenvectors. Clearly, if Eq. 19 is to be satisfied by any eigenvector assignment algorithm, W_{32} and W_{33} whether chosen or calculated must be such that:

$$\text{Range}\ [W_{32} \mid W_{33}] \subseteq \text{Range}\ [T_2] \tag{20}$$

Thus, if $T_2 = 0$, which is equivalent to requiring independent integral state feedback of each of the undisturbable state variables $x_1, x_2, \ldots, x_p$, $(p \le k)$, from Eq. 20 W_4^{-1} cannot exist. This is an algorithm-independent result and shows that any eigenvector assignment procedure would fail to obtain a matrix W of the required form in Eq. 18. With $T_2 \ne 0$, the result shows that with undisturbable x_i integral feedback of $(x_i + \sum_{j=p+1}^{n} x_j\alpha_j)$ (with $i\le p$ and at least one $\alpha_j \ne 0$) is possible. From a practical viewpoint, this is not as serious a disadvantage of the method as might first appear. For example, a proportional plus integral state feedback controller can certainly be designed, if some of the entries in H_{12} are made nonzero.

NUMERICAL EXAMPLE

A hypothetical state-space system was developed to illustrate the design procedure and several of the theoretical concepts. The form of the open-loop system is as defined in Eq. 1 and the coefficient matrices are:

$$A = \begin{bmatrix} -0.932 & 0.850 & -1.668 & 0.854 \\ -0.060 & -2.018 & 2.060 & -1.030 \\ -1.190 & 0.958 & -1.810 & 0.710 \\ -0.320 & 1.424 & -1.680 & 0.35 \end{bmatrix}$$

$$B = \begin{bmatrix} 5.30 & 1.60 & 1.96 \\ -3.50 & -2.0 & -1.20 \\ -0.50 & -1.50 & 0.60 \\ 1.0 & 0 & 1.80 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2.4 & -0.04 \\ -3 & 1.3 \\ 0 & 1.3 \\ 1.8 & 0 \end{bmatrix}$$

The eigenvalues of A are -3.0, -0.51, -1.0, and $+0.1$; hence, the system is open-loop unstable. Also, since A does not contain any zero elements, the conditions for undisturbability defined by Eqs. 5 and 7 are not met; hence, all four state variables are disturbable by either of the two disturbances variables. Since $y_2 = x_3$, the second output variable is disturbable. But, since y_1 is a linear combination of x_1 and x_2, it is not possible to draw any immediate conclusions as to whether y_1 is disturbable or not.

The control objectives were: a) to make the closed-loop system stable; b) to make y_1 and y_2 undisturbable by ξ_1 and ξ_2 (which will illustrate the design of a system for *output* undisturbability); and c) to specify complex eigenvalues and eigenvectors (to illustrate that they can be handled directly).

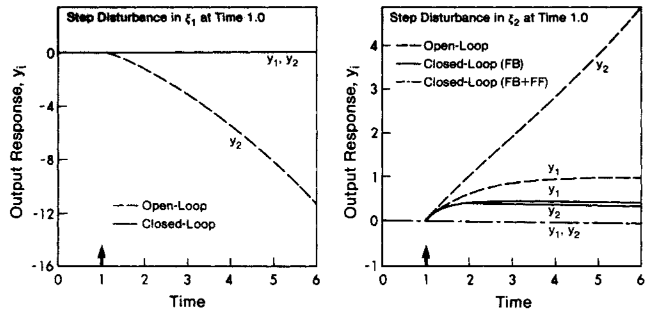

Figure 1. Simulated output responses of the open-loop and closed-loop system of example 1, to disturbances in ξ_1 (left) and ξ_2 (right).

Because the specification is for output undisturbability and y is not a subset of x, it is necessary to transform the system. The transformation (cf. Eq. 8) is:

$$z = Q^{-1}x = \left[\frac{C}{E}\right] x = \begin{bmatrix} 1 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \qquad (21)$$

and the coefficient matrices of the transformed system defined by Eq. 9 become:

$$Q^{-1}A\,Q \triangleq \tilde{A} = \begin{bmatrix} -0.98 & -0.02 & 0.02 & 0.03 \\ -1.19 & -1.81 & 1.91 & 0.71 \\ -0.06 & 2.06 & -1.97 & -1.03 \\ -0.32 & -1.68 & 1.68 & 0.35 \end{bmatrix} \qquad (22)$$

$$Q^{-1}D \triangleq \tilde{D} = \begin{bmatrix} 0 & 1 \\ 0 & 1.3 \\ -3 & 1.3 \\ 1.8 & 0 \end{bmatrix}$$

Comparison of the system characterized by $\{\tilde{A},\tilde{D}\}$ with the conditions in Eqs. 5 and 7 reveals that state variable $z_1 = y_1$ is "almost undisturbable" with respect to disturbance ξ_1, since $d_{11} = 0$ and $\tilde{a}_{12}$, $\tilde{a}_{13}$ and $\tilde{a}_{14}$ are very small relative to the other elements of A. However, a feedback controller will be required to generate the conditions necessary for complete output undisturbability. Since $\tilde{d}_{11}$ and $\tilde{d}_{21}$ are both zero, the condition implied by Eq. 7 is already met by $\tilde{d}_1$ and hence y_1 and y_2 can be made undisturbable by ξ_1 without the use of feedforward control. However, feedforward control will be required to give $\tilde{l}_2$ the proper structure to make the system undisturbable to ξ_2.

In the transformed system, the number of inputs is greater than specified number of undisturbable states. Therefore, it is possible to arbitrarily specify all the eigenvalues and three elements of each eigenvector as follows:

$$\Lambda = \text{diagonal}\ \{-2+j,\ -2-j,\ -3.0,\ -4.0\} \qquad (23)$$

$$\tilde{W} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1+j & 1-j & 0 & 0 \\ 0.5-j & 0.5+j & 1 & 1 \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix} \qquad (24)$$

Note that the unspecified elements w_{ij} are to be calculated as part of the design procedure and that the complex eigenvalues and elements of the eigenvectors must occur as complex conjugate pairs.

It is obvious that $\tilde{W}$ defined by Eq. 24 meets the condition for undisturbability specified by Eq. 6 and that the closed-loop system is stable. The feedback controller matrix is calculated as follows:

a) The 1×3 matrix $\tilde{W}_{21}$ and the 1×1 matrix $\tilde{W}_{22}$ are calculated by the procedure associated with Eqs. 11 through 14.

b) Since $\tilde{W}$, $\tilde{V}=\tilde{W}^{-1}$ and Λ are now completely specified, the feedback controller matrix $\tilde{K}$ can be calculated from Eq. 15.

The variables $z_1=y_1$ and $z_2=y_2$ of the closed-loop system are undisturbable by ξ_1 when this feedback controller is used, but a feedforward controller must be designed to make them undisturbable by ξ_2. In this example:

$$\tilde{l}_2 = \tilde{B}\,K^{FF} + \tilde{d}_2 \qquad (25)$$

If the choice $l_2^T = [0,0,1.3,\beta]$ is made where the number 1.3 is chosen arbitrarily, K^{FF} and β can be calculated from Eq. 25.

The feedback controller $\tilde{K}$ is then transformed to correspond to the original system by using the relationship $K=\tilde{K}\,Q^{-1}$ and the controller matrices are:

$$K = \begin{bmatrix} 0.13 & 0.40 & -1.31 & 0.99 \\ -0.45 & 0.51 & 1.61 & -0.86 \\ -2.36 & -2.64 & 4.29 & -2.50 \end{bmatrix} \qquad (26)$$

$$K^{FF} = [-0.12 \quad 0.63 \quad -0.69]^T$$

Simulated Results. Figure 1 shows the simulated response of y_1 and y_2 to a step change in ξ_1 under both open-loop and closed-loop conditions. As expected, the open-loop response of y_1 is essentially zero. The unstable open-loop response of y_2 is stabilized and made undisturbable by the feedback controller and the undisturbability of y_1 maintained. Note that the closed-loop system has zero offset even though it uses proportional feedback control only; it is subjected to a sustained step disturbance in ξ_1; and has no natural integrating modes.

Figure 1 also shows the simulated system responses to a step change in ξ_2. The open-loop responses of y_1 and y_2 are improved considerably by the feedback controller. But, as predicted by theory, both feedforward and feedback control are needed to make y_1 and y_2 undisturbable by ξ_2.

EVAPORATOR APPLICATION

The design procedure for undisturbability was evaluated experimentally by designing a controller for the computer-controlled, pilot-plant evaporator at the University of Alberta. This evaporator has been used previously to evaluate a number of modern multivariable control methods (Fisher and Seborg, 1976). Hence, it is possible to make direct comparisons between alternative control techniques.

A schematic diagram of the evaporator and the conventional multi-loop control scheme used as a basis of comparison is shown in Figure 2. The primary control objective is to keep the product concentration, $C2$, constant despite disturbances in the feed flow rate, F, the feed concentration, CF, and/or the feed enthalpy HF. It is also necessary to keep the two liquid holdups, $W1$ and $W2$, within operating limits, but small variations in these variables are acceptable. The control (manipulated) variables are the steam flow, S, and the bottom product flow rate from each of the two effects, $B1$ and $B2$. In summary, the

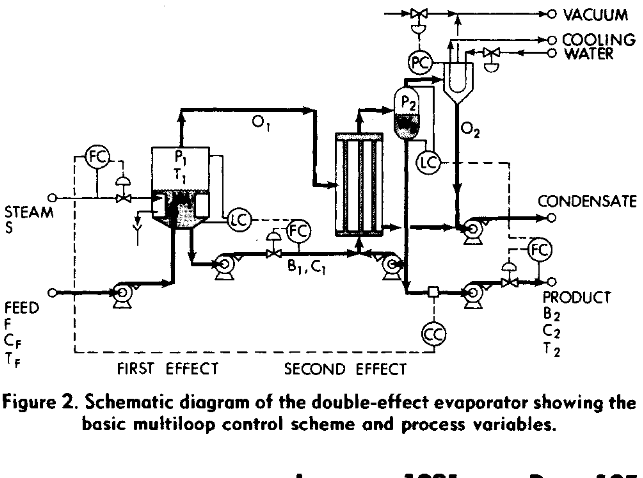

Figure 2. Schematic diagram of the double-effect evaporator showing the basic multiloop control scheme and process variables.

evaporator has three output variables, y, three control variables, u, and three disturbance variables, ξ.

Model. A number of different models of the evaporator have been derived in previous studies (e.g., Newell and Fisher, 1972). The model used in this study was the simplified, third-order, discrete, state-space model written as a function of normalized perturbation variables:

$$x(i+1) = \phi\, x(i) + \Delta\, u(i), + \theta\, \xi(i),\ y(i) = C\, x(i) \quad (27)$$

or

$$\begin{bmatrix} W1\,(i+1) \\ W2\,(i+1) \\ C2\,(i+1) \end{bmatrix} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 0.96 \end{bmatrix} \begin{bmatrix} W1\,(i) \\ W2\,(i) \\ C2\,(i) \end{bmatrix}$$

$$+ \begin{bmatrix} -0.0325 & -0.0811 & 0 \\ -0.0377 & 0.0854 & -0.0406 \\ 0.0527 & -0.0441 & 0 \end{bmatrix} \begin{bmatrix} S\,(i) \\ B1\,(i) \\ B2\,(i) \end{bmatrix} \quad (28)$$

$$+ \begin{bmatrix} 0.120 & 0 & -0.0135 \\ 0.0032 & 0 & -0.0156 \\ -0.0218 & 0.0398 & 0.0218 \end{bmatrix} \begin{bmatrix} F\,(i) \\ CF\,(i) \\ HF(i) \end{bmatrix}, \text{ and } C = I_3$$

The two unit eigenvalues in ϕ follow directly from the "integrating nature" of the two liquid holdups. This model is not as accurate as the 5^{th} and 10^{th} order models used in other studies, but is in reasonable agreement with experimental data and makes it much easier to illustrate the design procedure.

Comparison of the model in Eqs. 27 and 28 with the conditions given in Eqs. 5 and 7 show:

a) All three state variables, $C2$, $W1$, and $W2$, are disturbable by F and HF under open-loop conditions (because the elements in column 1 and 3 of θ are all nonzero).

b) $W1$ and $W2$ are undisturbable by CF (because $\phi_{13} = \phi_{23} = 0$ and $\theta_{12} = \theta_{22} = 0$).

c) $W1$ and $C2$ are undisturbable by $B2$ (because $\phi_{12} = \phi_{32} = 0$ and $\Delta_{13} = \Delta_{33} = 0$). This implies that $W1$ and $C2$ are *structurally uncontrollable* by $B2$ and indicates why, in the multiloop scheme in Figure 2, $W2$ is controlled by manipulating $B2$.

Controller. Previous operating experience has shown that the most frequent and severe disturbances in product concentration, $C2$, are produced by variations in feed flow rate, F. Therefore, for purposes of this example the design objectives for the controller were defined as: a) to make $C2$ undisturbable by F; b) to improve the speed of response by assigning all closed-loop eigenvalues closer to the origin; and c) to preserve the open-loop undisturbability of $W1$ and $W2$ by CF.

Since $\theta_{31} \neq 0$, it is not possible to make $C2$ undisturbable with respect to F by feedback control alone. However, in the discrete form of Eq. 3, $L = \theta + \Delta\, K^{FF}$. Hence, it is possible to use feedforward control to produce the required structure in L. Feedback control can then be used to maintain the desired structure in $H = \phi + \Delta\, K$. Thus, a control law of the form:

$$u(i) = Kx(i) + K^{FF}\, \xi_1(i) \quad (29)$$

is required. Setting $l_1^T = [\alpha, \beta, 0]$ with α and β arbitrarily chosen as 0.12 and 0.0032 respectively gives:

$$K^{FF} = \begin{bmatrix} 0.310 \\ -0.124 \\ -0.549 \end{bmatrix} \quad (30)$$

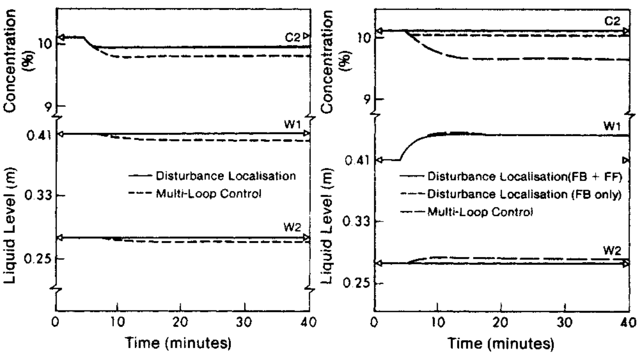

Figure 3. Simulated evaporator responses comparing disturbance localization controllers vs. conventional multiloop control. The undisturbability of $W1$ and $W2$ to a −30% change in feed composition is maintained (left) but feedback plus feedforward control is required to make $C2$ undisturbable to a +20% step in feed flow (right).

Note that if α and β were chosen as zero, all three state variables would be undisturbable with respect to F. This is possible because in this particular example Δ is nonsingular and consequently the equation, $l_1 = \theta_1 + \Delta K^{FF}$, has an exact solution for K^{FF}.

To complete the controller design, the feedback matrix K is calculated using the eigenvector/eigenvalue assignment procedure described previously. Since in this example the number of inputs, r, equals the number of states, n, it is possible to assign all elements of the eigenvector matrix W. If W is chosen to be a diagonal matrix, the following design objectives are realized.

i) $C2$ is undisturbable with respect to F since W and l_1 are in the form of Eqs. 6 and 7.

ii) $W1$ and $W2$ are still undisturbable with respect to CF (as was the case for the open-loop system in Eq. 28.

Choosing $W = I$, and assigning the closed-loop eigenvalues to be 0.28, 0.47, and 0.65 leads to the following state feedback control matrix:

$$K = \begin{bmatrix} 2.70 & 0 & -9.69 \\ 3.23 & 0 & 3.88 \\ 4.29 & 13.05 & 17.17 \end{bmatrix} \quad (31)$$

Simulation Results. Figure 3 compares the simulated response of the evaporator using disturbance localization vs. multiloop controllers as shown in Figure 2. In all cases, the disturbance localization controller gives better control.

The following controller gains give acceptable experimental responses. Hence, these values were used as the basis of comparison,

$$K = \begin{bmatrix} 0 & 0 & -4.98 \\ 3.52 & 0 & 0 \\ 0 & 15.8 & 0 \end{bmatrix} \quad (32)$$

Figure 3 (left) shows that $W1$ and $W2$ are undisturbable with respect to a −30% change in CF. Note that since the feed flowrate was held constant, the feedforward contribution is zero and the undisturbability of $W1$ and $W2$ is produced by feedback action alone. The controller was not designed to make $C2$ undisturbable by CF and an offset in $C2$ is evident.

Figure 3 (right) shows the response to a +20% step change in feed flow and it can be seen that the combined feedforward and feedback controller makes $C2$ undisturbable. If the feedforward control is omitted there is a small offset in $C2$, but the $W1$ and $W2$ responses remain unchanged.

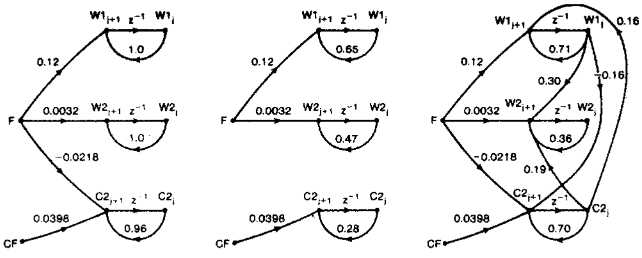

Figure 4. Signal flow graphs of the evaporator system under open-loop conditions (left); with a disturbance localization controller (center); and with conventional multiloop control (right).

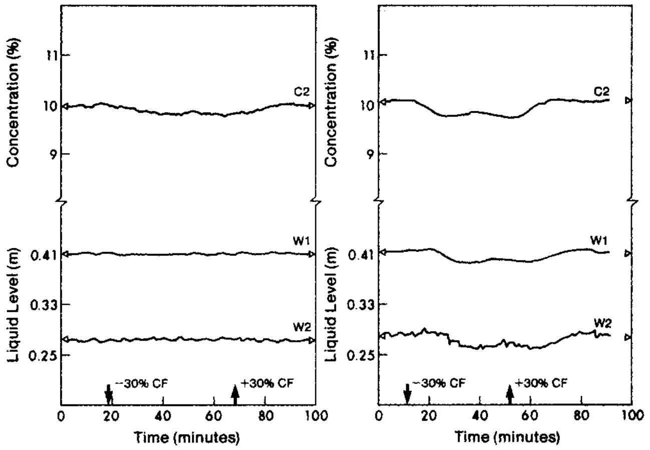

Figure 5. Experimental evaporator responses to ±30% step changes in feed composition (at times indicated by arrows). The undisturbability of *W*1 and *W*2 is maintained by the *FB* controller (left). (Right) regulation with multiloop control.

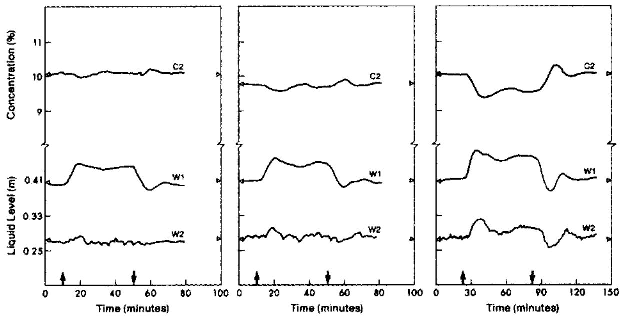

Figure 6. Experimental evaporator responses to ±20% step changes in feed flow (indicated by arrows). *FB* + *FF* controller designed to make C2 undisturbable (left). This is significantly better than *FB* alone (center) or multiloop control (right) (cf. Figure 3, right).

Signal Flow Analyses. The signal flow in Figure 4 (left) shows that in the third-order evaporator model, all three states are "accessible" by a disturbance in feed flow, *F*, but only the product concentration is "accessible" by a disturbance in feed concentration, *CF*. "Accessibility" of one variable by another means essentially that they are connected by a path which does not require going against the directions shown by the arrows on the signal flow graph. Nonaccessibility of x_i (or y_i) by ξ_j implies *undisturbability*. Undisturbability has also been defined in signal flow graph terminology (Shah et al., 1977).

Figure 4 (center) is the signal flow graph of the closed-loop evaporator system using the disturbance localization controller. Comparison with the open-loop system shows that the feedforward plus feedback controller eliminates the path from *F* to *C*2, and hence makes *C*2 inaccessible and undisturbable by *F*.

Figure 4 (right) is the signal flow graph of the evaporator system using the basic multiloop controller. Note that all the state variables become accessible to both disturbances, and that a number of interactions are introduced where none existed in the open-loop case.

Experimental Results. Figure 5 shows the response of the evaporator to two 30% step changes in feed concentration, introduced at the times indicated by the arrows on the time axis. The disturbance localization controller was designed to retain the open-loop undisturbability of the liquid levels to concentration changes and the results in Figure 5 (left) confirm that there is no significant effect of *CF* on *W*1 or *W*2. The controller was not designed to make *C*2 undisturbable by *CF* and a small offset in *C*2 is obvious in Figure 5 (left). (It is of interest to note that this controller is an "ideal" modal controller, since the open-loop and closed-loop eigenvectors are identical.)

The evaporator response to +30% changes in feed concentration using a multiloop controller are shown in Figure 5 (right). These experimental results confirm the conclusion derived from the simulation results (cf. Figure 3) that the controller designed for undisturbability gives better results than the conventional multiloop PID controller designed by on-line tuning.

Figure 6 shows the evaporator response to ±20% disturbances in feed flow when using feedforward plus feedback control (left), feedback control alone (center) and multiloop control (right).

The *FF*+*FB* controller used in Figure 6 (left) should make *C*2 undisturbable with respect to variations in feed flow. However, small variations are present and these can be attributed to modelling errors introduced during the derivation of the third-order model.

If the feedforward control action is omitted, control deteriorates as shown by Figure 6 (center) but is still better than that produced by multi-loop control. It should be noted that the multiloop results can be improved significantly by the addition of feedforward action. However, based on previous work it is unlikely that results would be as good as those in Figure 6 (left). The center and right portions of Figure 6 can be compared directly, since they both involve only feedback control.

In general, the design procedure for "undisturbability" was found to be practical and convenient. It was easy to use, gave considerable insight into system performance, placed the closed-loop eigenvalues and eigenvectors in the desired locations, produced practical controllers with reasonable gains, and performed well experimentally. To the best of our knowledge, this is the first reported experimental application of this design approach. In summary, the results are encouraging enough to suggest that industrial users should give serious consideration to this method, especially when there are particular state variable/disturbance pairs (e.g., x_i/ξ_j) that must be regulated.

NOTATION

A	= state coefficient matrix
B	= control or input coefficient matrix
B_1, B_2	= partitions of B (cf. Eq. 14)
B^*	= pseudoinverse of B
C	= output coefficient matrix
D	= disturbance or load coefficient matrix
E	= partition of a transformation matrix
$G_L(s)$	= load transfer function
H	= closed-loop system matrix (Eq. 3)
I	= identity matrix
J	= Jordan block
K	= state feedback control matrix
K^{FF}	= feedforward control matrix
k	= number of undisturbable state variables
L	= disturbance (or load) coefficient matrix for the closed-loop system with feedforward control
l_j	= j^{th} column of L
m	= dimension of output vector
n	= dimension of state vector
P	= constant coefficient matrix $(n-r)\times(n-r)$, as defined in Eq. 14

p	= number of state variables requiring integral feedback
Q	= transformation matrix as defined in Eq. 8
T	= matrix used to augment original system as defined in Eq. 16
q	= dimension of disturbance vector
r	= dimension of control vector
s	= Laplace operator
u	= control vector, $r\times 1$
V	= closed-loop system reciprocal (or left) eigenvector matrix ($V=W^{-1}$)
W	= closed-loop system eigenvector matrix
x	= state vector, $n\times 1$
x^{ξ}	= closed-loop response to disturbance ξ
y	= output vector, $m\times 1$
y^{ξ}	= output response of the closed-loop system to disturbance ξ

Greek Letters

ϕ	= state coefficient matrix for the discrete system
Δ	= control coefficient matrix for the discrete system
θ	= disturbance coefficient matrix for the discrete system
Λ	= $n\times n$ diagonal matrix whose diagonal elements are the eigenvalues of H
ξ	= disturbance vector, $q\times 1$

Superscripts

FF	= feedforward
-1	= matrix inverse
T	= transpose of a vector or matrix
*	= pseudoinverse of a matrix
~	= transformed matrix

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Manuscript received January 22, 1979; revision received May 15, and accepted May 20, 1980.

Correlation of Solubility of Hydrogen in Hydrocarbon Solvents

A correlation is developed for the solubility of hydrogen in hydrocarbon solvents at temperatures from 310 to 700 K and pressures to 30 MPa (300 bar). The fugacity of dissolved hydrogen at zero pressure is correlated as a function of solubility parameter and temperature. The high-pressure fugacity is obtained upon applying a Poynting factor for which the required partial molal volume of hydrogen is also correlated in terms of solubility parameter and temperature. The Henry constant of hydrogen is included in the correlation. The correlated results are compared with experimental data for 22 systems of binary and ternary solutions and five narrow-boiling cuts of coal liquids. The overall average absolute deviation for all systems studied is approximately 6%.

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SCOPE

Recent intensified development of hydrofining processes, particularly coal liquefaction, has increased the need for knowledge of hydrogen solubility in heavy hydrocarbons at elevated temperatures and pressures. This information is required for engineering design and analysis of reaction kinetics of hydrofining processes.

Hildebrand and Scott (1950) showed that gas solubility can be correlated with solubility parameter at room temperatures and low pressures. Chao and Seader (1961) and Grayson and Streed (1963) presented correlations for K-values of hydrogen with hydrocarbons, in which the regular solution-theory was

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0001-1541-81-3873-0138-$02.00.